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BIFURCATIONS AND POSTBUCKLING BEHAVIOR OF VIBRATING OF THE NONHOMOGENEOUS, NONLINEAR ELASTIC SYSTEM WITH MULTIPLE INDEPENDENT BIFURCATION PARAMETERS†

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Abstract—The asymptotic approach presupposing the use of the WKB method is employed to bifurcations and postbuckling behavior of vibrating of the nonhomogeneous elastic system with two independent bifurcation parameters. The extension to the more independent bifurcation parameters is a trivial matter. Guided by the principle of the WKB method devised originally for the linear problem of beams, plates and shells, the solution of some nonlinear problem of nonhomogeneous structures is considered.

1. INTRODUCTION

There are several problems that arise in the investigation of nonlinear phenomena which exhibit the property of bifurcation and postbuckling behavior of the nonhomogeneous system.

Especially important from the point of view of theoretical study and applications of these problems are the structures with variable rigidity or geometry and loaded by multiple independent bifurcation external forces. The latter is connected with the necessity to construct the fundamental characteristic or boundary surface (Papkovich, 1941; Huseyin, 1978) which separates the fundamental stable region from the unstable region of the structure.

It should be noted that some important results of the perturbed problem for a system with constant rigidity were obtained by Kolkka (1984) and Matkowsky and Reiss (1977).

In this paper, the asymptotic approach presupposing the use of the WKB method is employed to study bifurcations and postbuckling behavior of vibrations of the nonhomogeneous elastic system with two multiple independent bifurcation parameters.

Extension to three or more independent bifurcation parameters is a trivial matter as shown in Keller (1985).

In spite of the relative simplicity of the WKB or eikonal approximation, surprisingly little use of it has been made in problems of buckling and vibration of nonhomogeneous structures, as discussed in Gristchak (1979).

2. FORMULATION OF THE PROBLEM

Consider an imperfect elastic inhomogeneous vibrating shaft, loaded by a constant axial time-independent force P and rotating with a constant angular velocity ω_1.

The exact Bernoulli–Euler theory is employed and we seek dynamic equilibrium solutions. The system is considered conservative and the governing equations may be derived via a variational principle or simple equilibrium considerations. Either way, the governing equation is:

$$
\begin{aligned}
&y^{iv}(1-\dot{y}^2)^{-1}-4\dddot{y}\ddot{y}\dot{y}(1-\dot{y}^2)^{-2}+\ddot{y}^3(1+3\dot{y}^2)(1-\dot{y}^2)^{-3} \\
&\quad +\lambda_f f(x)\ddot{y}(1-\dot{y}^2)^{-3/2}+\mathring{\beta}\ddot{y}(1-\dot{y}^2)^{-1}+2\mathring{\beta}\dddot{y}(1-\dot{y}^2)^{-1}+3\beta\ddot{y}^2\dot{y}(1-\dot{y}^2)^{-2} \\
&\qquad +\lambda_f f(x)\delta\ddot{g}(x)-\lambda_r\cdot\psi(x)[y-\delta g(x)]+\tilde{\lambda}_v\phi(x)y_{,tt}=0 \qquad (1)
\end{aligned}
$$

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where $y = y(x)/l$ is the nondimensional transverse displacement; $x = x/l$ is the arclength;

$$\lambda_f = \frac{Pl^2}{B_0}; \quad B(x) = B_0[1+\tilde{B}(x)]$$

is the bending rigidity of the shaft;

$$f(x) = 1/1+\tilde{B}(x); \quad (\ )_{,t} = \frac{\mathrm{d}}{\mathrm{d}t}(\ ); \quad \lambda_r = \frac{\rho_0\omega_1^2 l^4}{B_0}; \quad \rho(x) = \rho_0[1+\tilde{\rho}(x)];$$

$$\psi(x) = \frac{1+\tilde{\rho}(x)}{1+\tilde{B}(x)}; \quad g(x) = \frac{Y_0(x)}{l}$$

is a function of the initial imperfections or the form of the initial stress-free wrinkling and $\delta > 0$ its amplitude, ρ_0 is the mass per unit length;

$$\ddot{\beta} = \frac{\ddot{B}(x)}{B(x)}; \quad \beta = \frac{\dot{B}(x)}{B(x)}; \quad (\dot{\ }) = \frac{\mathrm{d}}{\mathrm{d}x}(\ )$$

denotes differentiation with respect to x;

$$\tilde{\lambda}_v = \frac{\mu_0 F_0 l^2}{g_0 B_0}; \quad \phi(x) = \frac{[1+\tilde{\mu}(x)][1+\tilde{F}(x)]}{1+\tilde{B}(x)}; \quad \frac{\mu_0}{g_0}$$

is the mass density; and

$$F(x) = F_0(1+\tilde{F}(x)) \tag{2}$$

is the cross-sectional area of the shaft.

The magnitude of the imperfection is assumed to be small. With this assumption, retaining terms of almost third degree leads to the problem we solve:

$$y^{iv}(1+\dot{y}^2) - 4\dddot{y}\ddot{y}\dot{y} + \ddot{y}^3 + \ddot{\beta}\ddot{y}(1+\dot{y}^2) + 2\beta\dddot{y}(1+\dot{y}^2) + 3\beta\ddot{y}^2\dot{y} + \lambda_f f(x)\ddot{y}\left(1+\frac{3}{2}\dot{y}^2\right)$$
$$+\lambda_f f(x)\delta\ddot{g}(x) - \lambda_r\psi(x)[y+\delta g(x)] + \tilde{\lambda}_v\varphi(x)y_{,tt} = 0. \tag{3}$$

Or, in a more general form, (3) can be written as:

$$L[y, \lambda_f, \lambda_r, \tilde{\lambda}_v] = N[y], \tag{4}$$

where

$$L[y, \lambda_f, \lambda_r, \tilde{\lambda}_r] = y^{iv} + \ddot{\beta}\ddot{y} + 2\beta\dddot{y} + \lambda_f f(x)\ddot{y} - \lambda_r\psi(x)y + \tilde{\lambda}_v\varphi(x)y_{,tt} \tag{5}$$

is the linear operator; and

$$N[y] = -\frac{4\dddot{y}\ddot{y}\dot{y}}{(1+\dot{y}^2)} - \frac{\ddot{y}^3}{(1+\dot{y}^2)} - \frac{3\beta\ddot{y}^2\dot{y}}{(1+\dot{y}^2)} - \frac{\lambda_f f(x)\ddot{y}\dot{y}^2}{2(1+\dot{y}^2)} - \frac{\lambda_f f(x)\delta\ddot{g}(x)}{(1+\dot{y}^2)}$$
$$-\frac{\lambda_r\psi(x)y\dot{y}^2}{(1+\dot{y}^2)} + \frac{\lambda_r\psi(x)\delta g(x)}{(1+\dot{y}^2)} + \frac{\tilde{\lambda}_r\varphi(x)y_{,tt}\dot{y}^2}{(1+\dot{y}^2)} \tag{6}$$

is the nonlinear operator. The boundary conditions are:

$$M_B[y] = 0. \tag{7}$$

The solution of the governing eqn (3) is chosen to be in the form of periodic vibrations

$$y(x, t) = w(x) \cos \omega_2 t \tag{8}$$

where ω_2 is the frequency of natural vibrations of the shaft.

After substituting (8) into (3) we obtain:

$$w^{iv}[1+\dot{w}^2 \cos^2 \omega_2 t] - 4\dddot{w}\ddot{w}\dot{w}\cos^2 \omega_2 t + \ddot{w}^3 \cos^2 \omega_2 t + 2\beta\dddot{w}(1+\dot{w}^2\cos^2 \omega_2 t) + 3\beta\ddot{w}^2\dot{w}\cos^2 \omega_2 t + \lambda_f f(x)\ddot{w}\left(1+\frac{3}{2}\dot{w}^2\cos^2 \omega_2 t\right) + \lambda_f f(x)\delta\ddot{g}(x)\sec \omega_2 t - \lambda_r \psi(x) w - \lambda_r \psi(x)\delta g(x)\sec \omega_2 t - \lambda_v \varphi(x) w = 0, \tag{9}$$

where

$$\lambda_v = \tilde{\lambda}_v \cdot \omega_2^2. \tag{10}$$

The wrinkling distribution $g(x)$ is taken to be compatible with the boundary conditions (7) and to satisfy the necessary continuity requirements, so that $g(x)$ corresponds to the appropriate eigenfunction expansions.

3. THE BIFURCATION PROBLEM

The bifurcation problem is given by

$$G[w, \lambda_f, \lambda_r, \lambda_v, \delta] = 0 \tag{11}$$

$$G[w, \lambda_f, \lambda_r, \lambda_v, 0] \equiv F[w, \lambda_f, \lambda_r, \lambda_v] = 0, \tag{12}$$

where

$$G = w^{iv}(1+\dot{w}^2\cos^2 \omega_2 t) - 4\dddot{w}\ddot{w}\dot{w}\cos^2 \omega_2 t + \ddot{w}^3\cos^2 \omega_2 t + \beta\ddot{w}(1+\dot{w}^2\cos^2 \omega_2 t) + 2\beta\dddot{w}(1+\dot{w}^2\cos^2 \omega_2 t) + 3\beta\ddot{w}^2\dot{w}\cos^2 \omega_2 t + \lambda_f f(x)\ddot{w}\cdot\left[1+\frac{3}{2}\dot{w}^2\cos^2 \omega_2 t\right] + \lambda_f f(x)\delta\ddot{g}(x)\cos^2 \omega_2 t - [\lambda_r \psi(x) + \lambda_v \phi(x)]w - [\lambda_r \psi(x) + \lambda_v \varphi(x)]\delta g(x)\sec \omega_2 t = 0, \tag{13}$$

subject to the boundary conditions (7). Following Kolkka (1984) and Plaut (1979), we seek the asymptotic solutions of (13) via the double perturbation parameter expansion:

$$w(x) = \varepsilon w_1(x) + \frac{1}{2!}[\varepsilon^2 w_{11}(x) + 2\varepsilon\eta w_{12}(x) + \eta^2 w_{22}(x)] + \frac{1}{3!}[\varepsilon^3 w_{111}(x) + \eta^3 w_{222}(x) + 3\varepsilon\eta^2 w_{122}(x) + 3\eta\varepsilon^2 w_{211}(x)] + O(s^4) \tag{14a}$$

$$\lambda_f = \lambda_f^c + \lambda_1\varepsilon + \lambda_2\eta + \frac{1}{2!}(\lambda_{11}\varepsilon^2 + 2\lambda_{12}\varepsilon\eta + \lambda_{22}\eta^2) + O(s^3) \tag{14b}$$

$$\lambda_r = \lambda_r^c + \eta, \tag{14c}$$

where

$$s^2 = (\varepsilon+\eta)^2.$$

The perturbation parameter ε is defined by

$$\varepsilon^2 \equiv \langle w, w \rangle, \tag{15}$$

where $\langle \tilde{f}, \tilde{g} \rangle$ is an appropriate inner product, given by

$$\langle \tilde{f}, \tilde{g} \rangle \equiv \int_0^1 \tilde{f}(x)\tilde{g}(x)\,\mathrm{d}x. \tag{16}$$

The perturbation parameter η is defined by (14c). The leading order terms in (14b, c), $\lambda_f^c, \lambda_r^c, \lambda_v^*$ lie on a path

$$\Gamma(\lambda_f^c, \lambda_r^c, \lambda_v^*) = 0 \tag{17}$$

in the λ_f, λ_r, λ_v surface and Γ is determined by the linearization of (13). The linearization of (13) for a perfect system, which is

$$w^{iv}+2\beta\dddot{w}+\dot{\beta}\ddot{w}+\lambda_f f(x)\ddot{w}-[\lambda_r\psi(x)+\lambda_v\varphi(x)]w = 0, \tag{18}$$

together with boundary conditions (7), which will be taken to be understood from this point on, yield the path along which the bifurcation takes place.

For the solution of the equation with variable coefficients (18) we can introduce the natural large parameter $h = l/c$, where c is the size of the cross-section of the shaft, or another type of parameter. In this case eqn (18) looks like the equation with small parameter near high derivatives (Gristchak, 1979):

$$\left(\frac{1}{h}\right)^4[w^{iv}+2\beta\dddot{w}+\dot{\beta}\ddot{w}]+\left[\frac{1}{h}\right]^2\bar{\lambda}_f f(x)\ddot{w}-[\bar{\lambda}_r\psi(x)+\bar{\lambda}_v\phi(x)]w = 0, \tag{19}$$

where

$$\bar{\lambda}_f = \frac{P}{E\sqrt{\mathcal{T}_0}}, \quad \bar{\lambda}_r = \frac{\rho_0\omega_1^2}{E}, \quad \bar{\lambda}_v = \frac{\mu_0 F_0}{g_0 E l^2}. \tag{20}$$

In this problem the large parameter h^4 is

$$h^4 = \frac{l^4}{\mathcal{T}_0}.$$

In accordance with the WKB method (Steele, 1971; Keller, 1985), the solution of eqn (19) is chosen to be in the form:

$$w(x) = \exp\int_{x_0}^{x} \xi(x)\,\mathrm{d}x \tag{21}$$

where

$$\xi(x) = \sum_{i=0}^{\infty} h^{1-i}\xi_{i+1}(x).$$

Equation (19) is transformed into the following equivalent equation:

$$h^{-4}[\xi^4+6\xi^2\xi'+4\xi\xi''+3\xi'^2+\xi'''+2\beta(\xi''+3\xi\xi'+\xi^3)+\beta(\xi'+\xi^2)]$$
$$+h^{-2}[\bar{\lambda}_f f(x)(\eta'+\eta^2)]-[\bar{\lambda}_r\psi(x)+\bar{\lambda}_v\varphi(x)]=0. \quad (22)$$

The first approximation ξ_1 will be obtained after the necessary estimation of the order of the magnitude of the terms in eqn (22). The corresponding solution is

$$w_1(x)=\exp\int_{x_0}^{x} h\xi_1(x)\,dx \quad (23)$$

as for the approximation of the first order.

The function $\xi_1(x)$ is obtained from the equation

$$\xi_1^4+\bar{\lambda}_f f(x)\xi_1^2-[\bar{\lambda}_r\psi(x)+\bar{\lambda}_v\varphi(x)]=0. \quad (24)$$

From eqn (24) we have

$$\xi_1^2=-\frac{\bar{\lambda}_f f(x)}{2}\left[1\pm\sqrt{1+\frac{4}{\bar{\lambda}_f^2 f^2(x)}[\bar{\lambda}_r\psi(x)+\bar{\lambda}_v\varphi(x)]}\right] \quad (25)$$

or

$$\xi_1^2=-r_{1,2}, \quad (26)$$

where

$$r_1=\frac{\bar{\lambda}_f f(x)}{2}\left[1+\sqrt{1+\frac{4}{\bar{\lambda}_f^2 f^2(x)}[\bar{\lambda}_r\psi(x)+\bar{\lambda}_v\varphi(x)]}\right] \quad (27)$$

$$r_2=\frac{\bar{\lambda}_f f(x)}{2}\left[1-\sqrt{1+\frac{4}{\bar{\lambda}_f^2 f^2(x)}[\bar{\lambda}_r\psi(x)+\bar{\lambda}_v\varphi(x)]}\right]. \quad (28)$$

The solution of the initial eqn (19) in the first approximation is

$$w_1(x)=C_1\sin\bar{r}_1x+C_2\cos\bar{r}_1x+C_3\sin\bar{r}_2x+C_4\cos\bar{r}_2x,\quad \bar{r}=\int_{x_0}^{x} r\,dx. \quad (29)$$

The corresponding differential equation in the first WKB approximation is

$$w^{iv}+\lambda_F\ddot{w}-\lambda_{rv}w=0, \quad (30)$$

where

$$\lambda_F(x)=\bar{\lambda}_f h^2 f(x),\quad \lambda_{rv}(x)=h^4[\lambda_f\psi(x)+\bar{\lambda}_v\varphi(x)]. \quad (31)$$

The nontrivial eigenfunctions are

$$w_1(x)=\phi_n(x)\equiv\sqrt{2}\sin(n\pi x),\quad n=1,2,3,\ldots \quad (32)$$

for the case of simply supported ends.

Substitution of (29) into (7) gives the sequence of paths

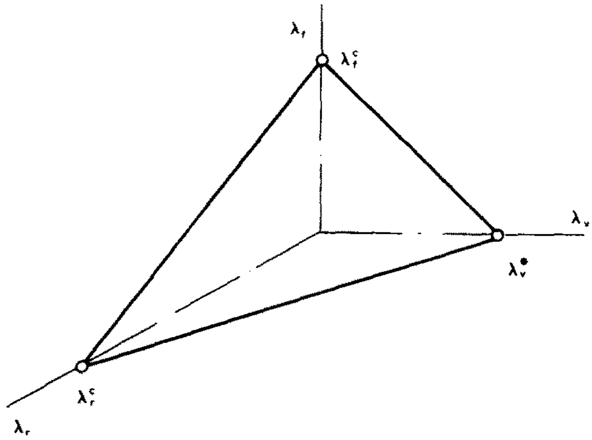

Fig. 1.

$$\Gamma_n(\lambda_f, \lambda_r, \lambda_v, n) = 0, \quad n = 1, 2, \dots \tag{33}$$

along which bifurcation takes place.

It is physically understandable that the critical integer n_{cr} is determined from

$$n_{cr} = \min_n \left[\min_{\lambda_f, \lambda_r, \lambda_v} \rho_n(\lambda_f, \lambda_r, \lambda_v) \right], \tag{34}$$

where $\rho_n(\lambda_f, \lambda_r, \lambda_v)$ is the distance from the origin of a point λ_f, λ_r, λ_v which lies on the path $\Gamma_n(\lambda_f, \lambda_v, \lambda_r)$.

In this case we found that $n_{cr} = 1$ and thus (Fig. 1)

$$\Gamma(\lambda_f^c, \lambda_r^c, \lambda_v^*) = \Gamma_1(\lambda_f^c, \lambda_r^c, \lambda_v^*) = 0. \tag{35}$$

Equation (35) represents a straight line between λ_f^c, λ_r^c and a straight line between λ_f^c, λ_v^* and λ_r^c, λ_v^* (Fig. 1) which intersects the axes at their respective "single" bifurcation points and the frequency of natural vibration of the system.

This type of boundary surface need not always be the case, however, especially when dealing with nonconservative problems (Huseyin, 1978); it depends on the changing of the wave numbers of the system in its postbuckling configuration.

Substitution of (14) into (13) leads to a sequence of linear boundary value problems to solve for the unknown $w_1(x)$, $w_{ij}(x)$, $w_{ijk}(x)$ and λ_i, λ_{ij}, $i, j, k = 1, 2, \dots$, etc.

The first-order equation satisfies

$$L(\lambda_f^c, \lambda_r^c, \lambda_v^*) w_1 = 0, \tag{36}$$

where $L(\lambda_f, \lambda_r, \lambda_v) W$ is defined by

$$L(\lambda_f, \lambda_r, \lambda_v) W = L_1(\lambda_f, \lambda_r) W + L_2(\lambda_v) W + O(s^4), \tag{37}$$

where $L_1(\lambda_f, \lambda_r) W$ is the linear differential operator which includes terms of magnitude up to the third order,

$$L_2(\lambda_v)W \equiv -\underset{\lambda_c^*}{\overset{\lambda_v}{\|}}\varphi(x)\left\{\varepsilon w_1+\frac{1}{2!}(\varepsilon^2 w_{11}+2\varepsilon\eta w_{12}+\eta^2 w_{22})\right.$$

$$\left.+\frac{1}{3!}(\varepsilon^3 w_{111}+\eta^3 w_{222}+3\varepsilon\eta^2 w_{122}+3\eta\varepsilon^2 w_{211})\right\}. \quad (38)$$

Collecting terms with the same power of ε, η and equating these terms to zero we obtain the system of the equations for the functions w_{ijk}:

$$\varepsilon: \quad w_1^{iv}+\ddot{\beta}w_1+2\beta\bar{\ddot{w}}_1+\lambda_f^c f(x)\ddot{w}_1-\lambda_r^c\psi(x)w_1-\lambda_v^*\varphi(x)w_1 = 0. \quad (39)$$

The term $\underline{\{\lambda_1 f(x)\delta\ddot{g}(x)\sec\omega_2 t\}}$ must be included in the discussion of the case of an imperfect beam:

$$L[\lambda_f^c,\lambda_r^c,\lambda_v^*]w_1 = 0 \quad (40)$$

$$\varepsilon^2: \quad w_{11}^{iv}+\ddot{\beta}\ddot{w}_{11}+2\beta\bar{\ddot{w}}_{11}+\lambda_f^c f(x)\ddot{w}_{11}-\lambda_r^c\psi(x)w_{11}-\lambda_v^*\varphi(x)w_{11} = -2\lambda_1 f(x)\ddot{w}_1;$$

$$\underline{\{-\tfrac{1}{2}\lambda_{11}f(x)\delta\ddot{g}(x)\sec\omega_2 t\}} \quad (41)$$

$$L[\lambda_f^c,\lambda_r^c,\lambda_v^*]w_{11} = -2\lambda_1 f(x)\ddot{w}_1$$

$$\varepsilon\eta: \quad w_{12}^{iv}+\ddot{\beta}\ddot{w}_{12}+2\beta\bar{\ddot{w}}_{12}+\lambda_f^c f(x)\ddot{w}_{12}-\lambda_r^c\psi(x)w_{12}-\lambda_v^*\varphi(x)w_{12} = \lambda_2 f(x)\ddot{w}_1+\psi(x)w_1;$$

$$\underline{\{-\lambda_f^c\lambda_{12}f(x)\delta\ddot{g}(x)\sec\omega_2 t\}} \quad (42)$$

$$L[\lambda_f^c,\lambda_r^c,\lambda_v^*]w_{12} = -\lambda_2 f(x)\ddot{w}_1+\psi(x)w_1$$

$$\eta^2: \quad w_{22}^{iv}+\ddot{\beta}\ddot{w}_{22}+2\beta\bar{\ddot{w}}_{22}+\lambda_f^c f(x)\ddot{w}_{22}-\lambda_r^c\psi(x)w_{22}-\lambda_v^*\varphi(x)w_{22} = 0;$$

$$\underline{\{\tfrac{1}{2}\lambda_{22}f(x)\delta\ddot{g}(x)\sec\omega_2 t\}} \quad (43)$$

$$L[\lambda_f^c,\lambda_r^c,\lambda_v^*]w_{22} = 0.$$

We see that (41), (42) are inhomogeneous forms of (40) evaluated at the singular point, and the Fredholm alternative requires that the inhomogeneous terms be orthogonal to all solutions of the corresponding homogeneous adjoint problem. The linear problem considered here is a self-adjoint one so we have

$$-2\lambda_1 f(x)(\ddot{\phi},\phi) = 0, \quad (44)$$

where

$$\phi(x) = w_1(x) \quad (45)$$

$$-2\lambda_1 f(x)(\ddot{\phi},\phi)+\psi(x)(\phi,\phi) = 0, \quad (46)$$

$$\lambda_1 = 0, \quad (47)$$

$$\lambda_2 = \frac{\psi(x)(\phi,\phi)}{f(x)(\ddot{\phi},\phi)} = \frac{\bar{\psi}(x)}{\mathscr{F}_0}, \quad (48)$$

where

$$\bar{\psi}(x) = \frac{\psi(x)}{f(x)}, \tag{49}$$

and $\mathscr{F}_0$ is the result from the Fredholm Alternative requirements. In the limiting case of a homogeneous system

$$[B(x) = \text{const}],\ \mathscr{F}_0 = -\pi^2 \quad \text{and} \quad \bar{\psi} = 1, \tag{50}$$

the same as for the first WKB approximation.

The problems for $w_{ij}(x)$ $(i, j = 1, 2)$ with (47), (48) become

$$L(\lambda_f^c, \lambda_r^c, \lambda_v^*) w_{ij} = 0, \tag{51}$$

so we take

$$w_{ij}(x) = 0, \quad i, j = 1, 2. \tag{52}$$

Proceeding in this recursive fashion, we obtain the third-order equations:

$$\begin{aligned}
\varepsilon^3:\quad & \overline{w_1^{iv}\dot{w}_1^2} + \frac{1}{3!} w_{111}^{iv} + 4\overline{\dddot{w}_1 \ddot{w}_1}\dot{w}_1 + \overline{\ddot{w}_1^3} + \ddot{\beta}\left(\overline{\ddot{w}_1 \dot{w}_1^2} + \frac{1}{3!}\ddot{w}_{111}\right) \\
& + 3\overline{\dot{\beta}\dot{w}_1}\ddot{w}_1^2 + 2\dot{\beta}\left(\overline{\dddot{w}_1 \dot{w}_1^2} + \frac{1}{3!}\dddot{w}_{111}\right) + \left(\frac{1}{3!}\lambda_f^c \ddot{w}_{111} + \frac{3}{2}\lambda_f^c \overline{\ddot{w}_1}\dot{w}_1^2\right. \\
& \left. + \frac{1}{2!}\lambda_1 \ddot{w}_{11} + \frac{1}{2!}\lambda_{11}\ddot{w}_1\right) f(x) - \frac{1}{3!}\lambda_r^c w_{111}\psi(x) - \frac{1}{3!}\lambda_v^* \varphi(x) w_{111} = 0
\end{aligned} \tag{53}$$

$$\begin{aligned}
& \frac{1}{3!} w_{1111}^{iv} + \frac{1}{3!}\ddot{\beta}\ddot{w}_{111} + \frac{1}{3!} 2\dot{\beta}\dddot{w}_{111} + \frac{1}{3!}\lambda_f^c \ddot{w}_{111} f(x) - \frac{1}{3!}\lambda_f^c w_{111}\psi(x) \\
& \quad - \frac{1}{3!}\lambda_v^* \varphi(x) w_{111} = -\frac{1}{2}\lambda_{11} f(x)\ddot{w}_1 - \overline{w_1^{iv}\dot{w}_1^2} 4\overline{\dddot{w}_1 \ddot{w}_1}\dot{w}_1 - \overline{\ddot{w}_1^3} - \frac{3}{2}\lambda_f^c \overline{\ddot{w}_1 \dot{w}_1^2} f(x) \\
& \quad - \ddot{\beta}\overline{\ddot{w}_1}\dot{w}_1^2 - 3\dot{\beta}\overline{\dot{w}_1 \ddot{w}_1^2} - 2\overline{\dot{\beta}\dddot{w}_1 \dot{w}_1^2} - \frac{1}{2!}\lambda_1 f(x)\ddot{w}_{11} = 0,
\end{aligned} \tag{54}$$

where

$$(\overline{\quad}) = (\quad)\cdot\cos^2\omega_2 t, \quad \lambda_1 = 0 \tag{55}$$

$$\begin{aligned}
L(\lambda_f^c, \lambda_r^c, \lambda_v^*) w_{111} = & -3\lambda_{11} f(x)\ddot{\phi} - (6\phi^{iv}\dot{\phi}^2 + 24\dddot{\phi}\ddot{\phi}\dot{\phi} + 6\ddot{\phi}^3 \\
& + 9\lambda_f^c \ddot{\phi}\dot{\phi}^2 \cdot f(x) + 6\ddot{\beta}\ddot{\phi}\dot{\phi}^2 + 18\dot{\beta}\dddot{\phi}\dot{\phi}^2 + 12\dot{\beta}\dddot{\phi}\dot{\phi}^2) E^*,
\end{aligned} \tag{56}$$

where

$$E^* = \cos^2\omega_2 t \tag{57}$$

$$\begin{aligned}
\eta^3:\quad & \frac{1}{3!}[w_{222}^{iv} + \ddot{\beta}\ddot{w}_{222} + 2\dot{\beta}\dddot{w}_{222} + \lambda_f^c f(x)\ddot{w}_{222} - \lambda_r^c \psi(x) w_{222} - \lambda_v^* \varphi(x) w_{222}] \\
& + \underbrace{\frac{1}{2!}\lambda_2 f(x)\ddot{w}_{22} - \frac{1}{2!}\varphi(x) w_{22}}_{=0} = 0
\end{aligned} \tag{58}$$

$$L(\lambda_f^c, \lambda_r^c, \lambda_v^*)w_{222} = 0 \tag{59}$$

$$\varepsilon\eta^2: \quad \frac{1}{2!}w_{122}^{iv} + \frac{1}{2!}\ddot{\beta}\ddot{w}_{122} + 2\frac{1}{2!}\dot{\beta}\dddot{w}_{122} + \frac{1}{2!}(\lambda_f^c f(x)\ddot{w}_{122} - \lambda_r^c\psi(x)w_{122} - \lambda_v^*\varphi(x)w_{122}) \tag{60}$$

$$+\frac{1}{2!}\lambda_{22}\ddot{w}_1 f(x) + \underbrace{\frac{1}{2!}2\lambda_2\ddot{w}_{12} - \frac{1}{2!}2\psi(x)w_{12}}_{=0} = 0 \tag{61}$$

$$L(\lambda_f^c, \lambda_r^c, \lambda_v^*)w_{122} = -\lambda_{22}f(x)\ddot{w}_1 \tag{62}$$

$$L(\lambda_f^c, \lambda_r^c, \lambda_v^*)w_{122} = -\lambda_{22}f(x)\ddot{\phi} \tag{63}$$

$$\eta\varepsilon^2: \quad \frac{1}{2!}(w_{211}^{iv} + \ddot{\beta}\ddot{w}_{211}\dddot{1}_{211} + \lambda_f^c f(x)\ddot{w}_{211} + 2\lambda_{12}f(x)\ddot{w}_1 - \lambda_r^c\psi(x)w_{211}$$

$$-\lambda_v^*\varphi(x)w_{211}) + \underbrace{\frac{1}{2!}\lambda_1 f(x)\cdot 2\ddot{w}_{12}}_{=0} + \underbrace{\frac{1}{2!}\lambda_2 f(x)\ddot{w}_{11} - \frac{1}{2!}\psi(x)w_{11}}_{=0} = 0 \tag{64}$$

$$L(\lambda_f^c, \lambda_r^c, \lambda_v^*)w_{211} = -2\lambda_{22}f(x)\ddot{w}_1 \tag{65}$$

$$L(\lambda_f^c, \lambda_r^c, \lambda_v^*)w_{211} = -2\lambda_{12}f(x)\ddot{\phi}. \tag{66}$$

Invoking the Fredholm alternative on (60), (63), (66) we obtain that

$$\lambda_{11} = \frac{2}{f(x)}\mathscr{F}_0^2 - \left[\frac{3}{2}\lambda_f^c\mathscr{F}_0 + \phi(\dot{\beta}, \ddot{\beta}, \mathscr{F}_0)\right]E^*(\omega_2, t), \tag{67}$$

where the function $\phi(\dot{\beta}, \ddot{\beta}, \mathscr{F}_0)$ depends on the solution $\phi(x) = w_1(x)$,

$$\lambda_{22} = 0, \quad \lambda_{12} = 0. \tag{68}$$

For the case of the homogeneous shaft, when $\phi(x) = w_1(x) = \sqrt{2}\sin\pi x$, we obtain:

$$\lambda_{11} = 2\pi^4 - \frac{3}{2}\lambda_f^c\pi^2E^*(\omega_2 t). \tag{69}$$

For another simple case of slowly changing functions of $B(x)$ the derivatives of $\beta(x)$ are considered to be equal to zero, i.e. $\dot{\beta}(x) = \ddot{\beta}(x) = 0$.

Therefore the solution is

$$\phi(x) = w_1(x) = \sqrt{2}\sin\pi x. \tag{70}$$

In this case we obtain

$$\lambda_{11} = \frac{2\pi^4}{f(x)} - \frac{3}{2}\lambda_f^c\pi^2E^*(\omega_2 t). \tag{71}$$

In the limit case of bifurcation of the shaft without vibration, we obtain the value of λ_{11}, which was derived by Kolkka (1984):

$$\lambda_{11} = 2\pi^4 - \frac{3}{2}\lambda_f^c\pi^2. \tag{72}$$

Higher order terms are calculated in exactly the same manner.

Summarizing results up to the present order we have:

$$w(x) = \varepsilon\phi(x) + O(s^3), \quad \text{and } w_{11} = w_{12} = w_{22} = 0; \tag{73}$$

$$\lambda_f = \lambda_f^c - \frac{\bar{\psi}(x)_\eta}{\mathscr{F}_0} + \left\{\frac{\mathscr{F}_0^2}{f(x)} - \left[\frac{3}{4}\lambda_f^c\mathscr{F}_0 + \frac{1}{2}\phi(\beta, \ddot{\beta}, \mathscr{F}_0)\right]E^*(\omega_2 t)\right\}\varepsilon^2 + O(s^3) \tag{74}$$

$$\lambda_r = \lambda_r^c + \eta. \tag{75}$$

The dependence of the amplitude ε on the parameters λ_f, λ_r can now be ascertained by simply eliminating η from (74), (75), resulting in:

$$\varepsilon^2\left\{\frac{\mathscr{F}_0^2}{f(x)}\left[\frac{3}{4}\lambda_f^c\mathscr{F}_0 + \frac{1}{2}\phi(\beta, \ddot{\beta}, \mathscr{F}_0)\right]E^*(\omega_2 t)\right\} = \lambda_f - \lambda_f^c + \frac{\bar{\psi}(x)}{\mathscr{F}_0}(\lambda_r - \lambda_r^c) \tag{76}$$

$$\varepsilon = \varepsilon(\lambda_f, \lambda_r, \omega_2) = \pm\left\{\frac{\mathscr{F}_0(\lambda_f - \lambda_f^c) + \bar{\psi}(x)(\lambda_r - \lambda_r^c)}{\mathscr{F}_0[4\mathscr{F}_0^2 - (3\lambda_f^c\mathscr{F}_0 + 2\phi)]f(x)E^*}4f(x)\right\}^{1/2}. \tag{77}$$

Here we note that

$$\mathscr{F}_0^2 - \left(\frac{3}{4}\lambda_f^c\mathscr{F}_0 + \frac{1}{2}\phi\right)f(x)E^*(\omega_2, t) > 0. \tag{78}$$

We also note that in the limit as either parameter approaches zero, we recover the individual single parameter bifurcations

$$\varepsilon(\lambda_f, \lambda_r) = \pm 2\sqrt{f(x)}\left\{\frac{\mathscr{F}_0(\lambda_f - \lambda_f^c) + \bar{\psi}(x)(\lambda_r - \lambda_r^c)}{\mathscr{F}_0[4\mathscr{F}_0^2 - (3\lambda_f^c\mathscr{F}_0 + 2\phi)f(x)]}\right\}^{1/2} \tag{79}$$

$$\varepsilon(\lambda_f, 0) = \pm 2\sqrt{f(x)}\left\{\frac{\mathscr{F}_0(\lambda_f - \lambda_f^c) + \bar{\psi}((x)\lambda_r^c)}{\mathscr{F}_0[4\mathscr{F}_0^2 - (3\lambda_f^c\mathscr{F}_0 + 2\phi)f(x)]}\right\}^{1/2} \tag{80}$$

$$\varepsilon(\lambda_f, 0, \omega_2) = \pm 2\sqrt{f(x)}\left\{\frac{\mathscr{F}_0(\lambda_f - \lambda_f^c) - \bar{\psi}(x)\lambda_r^c}{\mathscr{F}_0[4\mathscr{F}_0^2 - (3\lambda_f^c\mathscr{F}_0 + 2\phi)]f(x)E^*(\omega_2 t)}\right\}^{1/2} \tag{81}$$

$$\varepsilon(0, \lambda_r, 0) = \pm 2\sqrt{f(x)}\left\{\frac{\bar{\psi}(x)(\lambda_r - \lambda_r^c) - \mathscr{F}_0\lambda_f^c}{\mathscr{F}_0[4\mathscr{F}_0^2 - 2\phi\cdot f(x)]}\right\}^{1/2} \tag{82}$$

$$\varepsilon(0, \lambda r, \omega_2) = \pm 2\sqrt{f(x)}\left\{\frac{\bar{\psi}(x)(\lambda_r - \lambda_r^c) - \mathscr{F}_0\lambda_f^c}{\mathscr{F}_0[4\mathscr{F}_0^2 - 2\phi f(x)E^*(\omega_2 t)]}\right\}^{1/2}. \tag{83}$$

The response diagram for the case $\lambda_v = 0$ is represented in Fig. 2 (Kolkka, 1984). For the case of a homogeneous system we obtain:

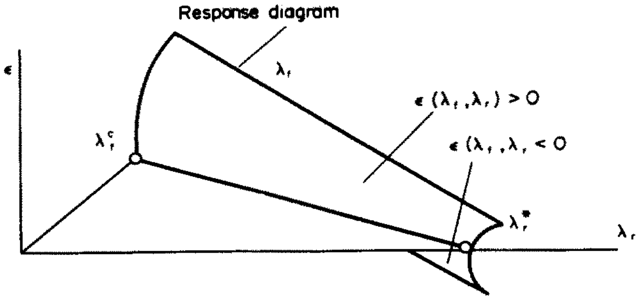

Fig. 2.

$$f(x) = 1, \psi(x) = 1, \lambda_f^c = \Pi^2, \lambda_r^c = \Pi^4; \quad \mathscr{F}_0 = -\Pi^2 \tag{84}$$

$$\varepsilon(\lambda_f, 0) = \frac{\pm 2}{\Pi^2}(\lambda_f - \Pi^2)^{1/2}, \tag{85}$$

$$\varepsilon(0, \lambda_r) = \frac{\pm 1}{\Pi^3}(\lambda_r - \Pi^4)^{1/2}. \tag{86}$$

Thus the problem (59)–(66) exhibits a supercritical bifurcation from the path (35).

The outer expansions for an imperfect system and the inner expansions in the vicinity of the critical path using a double small parameter expansion are calculated in the same manner as in Kolkka (1984).

4. CONCLUDING REMARKS

Guided by the principle of the WKB method devised originally for linear problems of beams, plates and shells, we considered how to solve some nonlinear problems of nonhomogeneous structures. It is worthwhile mentioning that the present paper gives a survey of a small part of the vast area where the WKB method would be applicable, at least in principle.

We note here that there is a practical problem in obtaining computer-calculated solutions of the nonlinear partial differential equation with variable coefficients and with small parameters near the highest derivative. As the parameter h^{-1} takes on smaller and smaller values, oscillations in the solutions become more closely spaced and it becomes necessary to introduce additional space steps to accurately describe the solution. Coupled with such factors as numerical accuracy and stability, this places a severe restriction on the total evolution time of the solution that can be computed.

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